Analytic Platforms in Cruising Aircraft

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An extensive cruise error analysis is presented for a strapdown inertial aircraft navigation system with pulse torqued instruments, in the presence of small amplitude oscillations. The analysis accounts for complex lateral and angular vibration with realistic models of coupled motion. Steady and secularly increasing position uncertainties, with accompanying Schuler oscillations, are expressed directly in terms of familiar system constants and statistical motion parameters.

Nomenclature

$\mathbf{a}$ = vibratory translational acceleration vector, g
a_j = jth component of $\mathbf{a}$, g
$\mathbf{A}$ = total nongravitational translational acceleration vector, g
$\tilde{\mathbf{A}}$ = uncertainty in $\mathbf{A}$, g
$[B]$ = orthogonal transformation from aircraft to inertial co-ordinates
$[\tilde{B}]$ = uncertainty in [B]
$[\tilde{B}_0]$ = initial attitude uncertainty matrix
$[C]$ = orthogonal transformation from aircraft to local co-ordinates
$[D]$ = orthogonal transformation from local to inertial co-ordinates
$\mathbf{F}$ = steady nongravitational translational acceleration vector, g
F_x = average component of $\mathbf{F}$ along roll axis, g
F_z = component of $\mathbf{F}$ along yaw axis, g
F_a = center frequency of translational vibration spectrum, cps
F_ω = center frequency of rotational vibration spectrum, cps
g = gravitational acceleration magnitude, 1 g
$\mathbf{G}$ = gravity vector in local coordinates
$[I]$ = 3×3 identity matrix
$\langle n_j \rangle$ = net drift rate in jth channel, rad/sec
n_{dj} = drift bias in jth channel, rad/sec
$[N]$ = skew-symmetric matrix of net drift rates, rad/sec
$[P]$ = offset matrix between local coordinates and mean positions of aircraft axes
$\mathbf{R}$ = geocentric position vector of aircraft, units implied in context
t = time, sec
W = schuler rate, units implied in context
ΔX = range navigation error, units implied in context
ΔY = cross-range navigation error, units implied in context
α_{yj} = output axis component of jth accelerometer misalignment, rad
α_{zj} = pendulous axis component of jth accelerometer misalignment, rad
β_{xz} = accelerometer vibropendulosity coefficient, g^{-1}
β_y = ratio of output axis inertia to accelerometer pendulosity, g-sec^2
γ_{xj} = input axis mass unbalance coefficient of jth gyro, g^{-1} sec^{-1}
γ_{zj} = spin axis mass unbalance coefficient of jth gyro, g^{-1} sec^{-1}
γ_{xz} = gyro anisoelastic parameter, g^{-2} sec^{-1}
γ_y = ratio of gyro output axis inertia to angular momentum, sec
θ = pulse torqued gyro resolution, rad
λ_j = initial attitude uncertainty about jth aircraft axis, rad
μ_j = fixed scale factor error of jth accelerometer
ν_j = bias error of jth accelerometer, g
$\rho_{\omega ij}$ = correlation between angular oscillation rates about (i) and (j) aircraft axes
τ_j = delay time associated with jth gyro loop, sec
φ_y = steady pitch displacement off vertical, rad
φ_z = steady yaw displacement, rad
ω_j = oscillatory angular rate about jth aircraft axis, rad/sec
ω_{jr} = rms value of ω_j, rad/sec

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Introduction

An extensive error analysis is presented herein, whereby the performance of a strapdown inertial navigation system is determined for cruising aircraft. The system envisioned consists of three pulse rebalanced single-degree-of-freedom rate integrating gyros, three pulse rebalanced single-axis pendulous integrating linear accelerometers, a digital differential analyzer (DDA) for performing all attitude update computations and co-ordinate transformations of measured velocity increments, plus all standard auxiliary equipment required to complete the navigation and flight control functions. Of particular importance in the analysis are the long-term effects of the various instrument and computation errors, which have not previously been evaluated for gimballess systems. To determine these effects, extensive use is made of separate analyses for instrument errors[1] and the propagation of these plus other (computation) errors in the attitude reference[2]; the present analysis presupposes a thorough acquaintance with both of these previous studies.

To place the techniques employed here into proper perspective, it is noted that the present goal is to introduce the application of powerful analytical tools to the evaluation of strapdown navigation. Initially this necessitates visualization of complex analytical expressions, frequently in combinations of double, triple, or higher order products (simply because navigation error terms are formed initially by multiplying components of complex linear and angular motion). At the same time, the desired outcome is a set of straightforward expressions which, despite their simplicity, provide a realistic measure of navigation performance. To achieve this ambitious result with a minimum of distracting algebraic manipulations, it is essential that all permissible simplifications be adopted at the outset. Specifically,

1) For a clear indication of strapdown navigation accuracy, only unaided systems are considered, e.g., no external doppler information is assumed present. Damping can be taken into account in a separate analysis.

2) Mechanization details that do not appreciably affect performance can be ignored. For this reason, a homogeneous spherical earth is entirely adequate for the gravitational field model (Ref. 3, p. 164). Also, where deterministic or measureable error sources are compensated through computer and/or instrument corrections, the error coefficients in the present analysis are merely redefined as the uncompensated residual portion.

3) Only the horizontal navigation accuracy need be analyzed. Vertical channel information is assumed to be provided by an ideal altimeter (Ref. 3, p. 165).

4) Typical 2-hr cruise trajectories have a net angular excursion in the vicinity of 30 deg, relative to inertial space. It is common practice (Ref. 3, p. 169) to limit the analysis to 2 hours maximum flight duration since this condition allows

an important simplification of the error vector differential equation. (The actual differential equation with time-varying coefficients is replaced by an approximate differential equation with constant coefficients). The analysis then leads to an approximate navigation error growth rate (naut miles/hr) which is a useful performance criterion for gimbaled and strapdown systems alike.

5) For illustrative purposes the analysis can be oriented toward certain typical quantitative conditions. For example, present undamped inertial systems are capable of accuracies on the order of a few naut miles/hr. In addition, angular oscillations are characterized as having amplitudes under 1° and mean frequencies (F_ω) near 10 rad/sec or greater. The concept of center frequency for angular oscillations (F_ω) and linear vibrations (F_a) is used somewhat freely, as if it were nearly the same in all three aircraft axes (or, rather, if all terms divided by it were diminished by the same order of magnitude). Also, linear accelerations due to vibration are assumed to be comparable with the resultant magnitude of lift and thrust. Significant departures from these conditions will not necessarily invalidate the results, but additional investigations may be required.

6) The flight is separated into phases during which the steady acceleration is essentially constant and the oscillatory motion statistics are erogodic.

7) To facilitate the formulation of products of lengthy trigonometric series, an analytical tool is borrowed from conventional celestial mechanics: long-range effects of interacting factors are isolated by retaining in the final product only the major terms (e.g., purely oscillatory errors are dropped if constant terms of comparable magnitude are present). Questions regarding rigorous demonstration of series convergence are not considered; justification of the analysis depends instead upon comparison with practical experience.

8) In line with the preceding item, the implications of random terms in the forcing function are not scrutinized. Although randomness in terms of a differential equation could in general have many ramifications, it will be noted that the deterministic components of the forcing function are much more effective here. Again, justification depends upon practice rather than mathematical rigor.

9) Several minor error sources have been excluded. Among these are[2] truncation and roundoff, which can be rendered negligible through proper computational design; fixed uniform information delays, including the expected (mean) delay due to quantization; limit cycling and random drifts; and the interaction of the cruise profile rate (e.g., 15°/hr) with gyro misalignment and scale factor errors.

It is noted that these nine conditions represent commonly accepted procedures and all simplifications introduced in the next section are covered thereby. In contrast to the last of the preceding items, those error sources which have been evaluated are: computation error—commutation (the result of quantizing finite rotations which are noncommutative); gyro imperfections—scale factor uncertainties, plus drift due to misalignment, null bias, mass unbalance, anisoelasticity, torque loop mismatch, angular acceleration about the output axis, gimbal displacement, and anisoinertia; and accelerometer imperfections—scale factor uncertainties, delay errors (including random delay due to quantization) plus additive errors due to misalignment, null bias, output axis motion, pendulous offset, anisoinertia, and physical separation.

In regard to physical separation, it is noted that a noncoincident triad of instruments will measure components of motion existing at three different points. Because the angular rate vector is the same at every point in a rigid body, physical separation of gyros will not present any inherent problems. Effects of accelerometer separation, however, must be taken into account. These effects are treated herein as equivalent additive instrument errors.

The complexity of the applicable linear and angular oscillatory motions in no way precludes a straightforward and accurate evaluation of navigation system performance. All that must be known about these oscillations, in the most general case, is the following: 1) mean and rms values of roll, pitch, and yaw angular rates, plus the coefficient of linear correlation between each pair; 2) mean and rms values of translational acceleration along the roll, pitch, and yaw axes, plus the coefficient of linear correlation between each pair; 3) coefficients of linear correlation for every combination of instantaneous vehicle rate $\omega_j(t)$ and delayed vehicle rate $\omega_k(t - 1/4F_\omega)$ about a perpendicular axis, where F_ω is the center frequency of the angular vibration spectrum; and 4) coefficients of linear correlation for every combination of instantaneous and delayed acceleration components, similar to those defined in item 3.

It was found that, for a typical set of system coefficients and motion parameters, correlation between linear and angular oscillations will not greatly influence the results. (Although this correlation could conceivably produce appreciable errors, these over overshadowed by other, larger sources.) Furthermore, the coefficients defined in items 3 and 4 are negligible in cases where no appreciable amount of unidirectional coning (defined in Ref. 2) or unidirectional cylindrical motion is present.

From the preceding discussion it is seen that random vibratory motions must be characterized for strapdown analysis in more detail than previously required for platform systems. In particular, the correlation between roll and yaw rates will directly influence the component of navigation error rate caused by noncommutativity.

In addition to the commutation error, major drift sources are gyro null bias, sensitivity to output axis angular acceleration, and anisoelasticity. Collectively, these errors generate a secularly increasing attitude uncertainty which interacts with the lift force to produce the major components of navigation error. Accuracies of a few naut miles/hr are possible at present; a goal of 1 mph may be realized in the not too distant future.

Analysis

A basic difference between the system of interest here and a Schuler tuned platform is the significance of the difference in orientation of sensor reference input axes from the local coordinates. In a Schuler tuned platform this difference corresponds to the attitude error; in the strapdown system a distinction must be made between this angular displacement and the attitude uncertainty. It is, therefore, necessary to derive the navigation error expressions in a formulation that accounts for this distinction.

This section contains an analysis whereby the effects of each error source upon navigation performance can be clearly expressed. To maintain the general scope of the study, the exact manner in which the separate error sources combine (which depends partially upon specific motion patterns) is not treated in detail. For brevity, rigorous definitions and detailed explanations are avoided here whenever it is possible to invoke known principles. Reference 3 (pp. 160–175) contains much information which, although derived for platform analyses, can be applied here also. At the outset the following will be noted.

The analysis deals with navigation over a spherical earth (Ref. 3, p. 164) to evaluate the accuracy of the horizontal channels (Ref. 3, p. 165) in an undamped inertial system during a short (Ref. 3, p. 169) cruise (2 hours or less). Equation (7.10) of Ref. 3 with the necessary typographical correction, would read as follows:

$$\dot{\boldsymbol{\psi}} + \boldsymbol{\omega} \times \boldsymbol{\psi} \doteq \boldsymbol{\epsilon} \tag{1}$$

where $\boldsymbol{\epsilon}$ and $\boldsymbol{\psi}$ represent the gyro drift and the angular error vectors, respectively. For strapdown analysis, Eq. (7) of Ref. 2 is a modification of this expression, in which the angular error derivative $\mathbf{N}$ is written explicitly in terms of the total drift (including gyro and computation errors) and its time integral $\boldsymbol{\xi}$:

$$\mathbf{N} \doteq \dot{\boldsymbol{\xi}} - \boldsymbol{\omega} \times \boldsymbol{\xi} \tag{2}$$

Another important difference between the preceding equations is the significance of $\boldsymbol{\omega}$; only the profile is involved in the former, whereas Eq. (2) utilizes the total inertial angular rate, including oscillations.

The model used here leads to an approximate navigation error differential equation, which will provide quantitative criteria of the type used for platform performance evaulation. The forcing function in the differential equation arises from acceleration measurement errors and interaction of attitude uncertainty (i.e., the time integral of $\mathbf{N}$ in the preceding equation) with the true acceleration.

The acceleration measurement errors mentioned previously are functions of components and products of components of lateral and angular motion.[1] The same is true of angular rate errors[1,2] but these are (at least potentially) more serious since, as previously mentioned, their time integrals appear in the navigation error forcing function. It follows that the total forcing function includes constants, terms which increase with time, and oscillatory terms of comparable magnitude.

Since only the long-range cumulative error is of interest here, a host of relatively ineffective components can be ignored in the final expressions for the forcing function. By definition, the error accumulated over a long duration is indistinguishable from the final error which would have resulted if the growth rate had been fixed at its average value. For a flight phase with erogodic motion properties, these time averaged quantities are immediately expressible as familiar statistical moments. Thus the long-range effects of the various error sources can be determined merely by inspecting the form and the coefficient magnitudes of the terms described in the preceding paragraph.

Preparation is now complete for implementing the analytical approach. To begin the error analysis, the following Cartesian coordinate sets are defined: 1) an orthogonal triad consisting of the roll x, pitch y, and yaw z axes; 2) an orthogonal triad of local coordinates, made up of the unit normal to the cruise trajectory plane y, the local vertical z, and their cross product x; and 3) an arbitrary inertial reference. The transformations from vehicle to inertial, vehicle to local, and local to inertial coordinates are expressed as $[B]$, $[C]$, and $[D]$, respectively; it follows that $[B] = [D]\,[C]$. Denoting the total nongravitational acceleration vector in vehicle coordinates as $(\mathbf{F} + \mathbf{a} = \mathbf{A})$ and the known gravity vector in local coordinates as

$$\mathbf{G} = \hat{\mathbf{G}} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{3}$$

the vectors representing actual and observed total acceleration can be written as

$$\ddot{\mathbf{R}} = [B]\mathbf{A} + [D]\mathbf{G} \qquad \hat{\ddot{\mathbf{R}}} = [\hat{B}]\hat{\mathbf{A}} + [\hat{D}]\mathbf{G} \tag{4}$$

respectively, where $\mathbf{R}$ is the aircraft position vector, with respect to the geocenter, and the circumflex denotes observed quantities throughout. These are related to corresponding true values and errors by

$$\hat{\mathbf{A}} = \mathbf{A} + \tilde{\mathbf{A}} \quad [\hat{B}] \doteq [B][I + \tilde{B}] \quad [\hat{D}] \doteq [D][I + \tilde{D}] \tag{5}$$

where $[I]$ is a 3×3 identity matrix and the tilde is used to denote uncertainty. When the preceding two relations are combined with second-order errors neglected

$$\tilde{\ddot{\mathbf{R}}} \triangleq \hat{\ddot{\mathbf{R}}} - \ddot{\mathbf{R}} = [D]\{[C]([\tilde{B}]\mathbf{A} + \tilde{\mathbf{A}}) + [\tilde{D}]\mathbf{G}\} \tag{6}$$

The usual procedure at this point in platform analysis is to invoke the condition that the total flight curvature, from the combined result of earth rate and motion along the cruise trajectory, is not much greater than 30 deg. It is then permissible to transform the preceding error vector into local coordinates,

$$\Delta\ddot{\mathbf{R}} \triangleq [D]^{-1}\tilde{\ddot{\mathbf{R}}} = [C]([\tilde{B}]\mathbf{A} + \tilde{\mathbf{A}}) + [\tilde{D}]\mathbf{G} \tag{7}$$

to express the horizontal components of $[\tilde{D}]\mathbf{G}$ to first order in terms of known geocentric distance ($|\mathbf{R}| = R$),

$$[\tilde{D}]\mathbf{G} = (-g/R)\begin{bmatrix} \Delta X \\ \Delta Y \\ 0 \end{bmatrix} \tag{8}$$

and to rewrite (7) as a second-order vector differential equation with constant coefficients,

$$\Delta\ddot{\mathbf{R}} + (g/R)\Delta\mathbf{R} = [C]([\tilde{B}]\mathbf{A} + \tilde{\mathbf{A}}) \tag{9}$$

With moderate flight path curvature, the solution to this equation will provide an accurate performance estimate for the horizontal navigation channels. Although the standard assumption of a perfect altimeter ($\Delta z \equiv 0$, and known R) seems at first to be contradicted by the presence of a three-dimensional forcing function, the vertical component of the analytical result will be ignored. With this understanding it is possible to continue the analysis in a straightforward manner without introducing special notation, partitioned matrices, etc.

The Schuler oscillation in the navigation error is abundantly clear from the homogeneous part of (9); the magnitude of these errors must be found in terms of the initial conditions and the forcing function. To this end the aircraft motion is described as follows: the vector $\mathbf{A}$, in vehicle coordinates as previously defined, consists of a zero mean Gaussian random vector ($\mathbf{a}$) and a fixed vector $\mathbf{F}$ made up of a net thrust F_x along roll and a lift F_z of essentially 1 g along yaw. The matrix $[C]$ is characterized by the relation

$$[C] = [P]\{[I] + [Q]\} \tag{10}$$

where $[P]$ represents a fixed angular displacement arising from a steady component of crosswind and/or angle of attack, and $[Q]$ for present purposes is adequately described in terms of oscillatory vehicle rate ($\boldsymbol{\omega}$) integrals as

$$[Q] \doteq \begin{bmatrix} 0 & -\int\omega_3 dt & \int\omega_2 dt \\ \int\omega_3 dt & 0 & -\int\omega_1 dt \\ -\int\omega_2 dt & \int\omega_1 dt & 0 \end{bmatrix} \tag{11}$$

Implicit in this approximation is the previously mentioned assumption that the peak angular displacements associated with the oscillatory vehicle rates (ω_1, ω_2, ω_3) are on the order of a degree. This formulation neglects only the second and higher order directional variations of the acceleration error under the action of angular oscillations about more than one aircraft axes. (An analysis including these terms would readily show that they are negligible, under the condition of small angular oscillation amplitudes stated previously. The contribution of yaw, roll, and pitch rate interactions to the generation of acceleration errors will arise from the attitude uncertainty matrix $[\tilde{B}]$.) The positive elements of $[Q]$ then constitute a Gaussian random vector with zero mean, which can be derived from the random oscillation rate vector $\boldsymbol{\omega}$ by a linear transformation, i.e., integration.

Because this random vehicle motion must be taken into account in any realistic performance analysis, it is immediately apparent that the forcing function in (9) must be considered at first in statistical terms. Its evaluation is further compounded by the dependence of each element in $\tilde{\mathbf{A}}$ and $[\tilde{B}]$ upon various components, and products of components, of $\mathbf{A}$ and $\boldsymbol{\omega}$. It might therefore be expected that a completely rigorous analysis, individually forming every conceivable product implicit in (9), would be quite lengthy. Fortunately, however, the expression can be significantly reduced while still in matrix form merely by examining the nature of the various matrix and vector product terms. This procedure begins with a further characterization of the components of $\mathbf{a}$ and $\boldsymbol{\omega}$ as noisy waveforms which can be represented by general Fourier series,

$$a_i = \sum_j K_{aij} \cos(2\pi f_{aij} t + \psi_{aij})$$
$$\omega_i = \sum_j K_{\omega ij} \cos(2\pi f_{\omega ij}\, t + \psi_{\omega ij}) \qquad (12)$$

(These will be referred to as series of the type S_{a1} and $S_{\omega 1}$, respectively.) The fact that both $\mathbf{a}$ and $\boldsymbol{\omega}$ are restricted to small-amplitude oscillations will guarantee that the bandwidth corresponding to each waveform is small in comparison with its center frequency. It follows that

$$\int_0^t \omega_j(\tau) d\tau \doteq \left(\frac{1}{2\pi F_\omega}\right) \omega_j \left(t - \frac{1}{4F_\omega}\right)$$
$$\frac{d\omega_j(t)}{dt} \doteq 2\pi F_\omega \omega_j \left(t + \frac{1}{4F_\omega}\right) \qquad (13)$$

(where the center frequency F_ω is expressed in cps) and similar relations hold for the integrals and derivatives of translational acceleration components. The characterization of random motions by (12) also leads to a simple formulation for products of vibration components. A term of the type $(\omega_i \omega_j)$ is equivalent to the sum

$$\omega_i \omega_j = \rho_{\omega ij} \omega_{ir} \omega_{jr} + S_{\omega 2} \qquad (14)$$

where ω_{ir}, ω_{jr}, $\rho_{\omega ij}$ represent the rms vibratory angular rates in the (i, j) channels and the corresponding linear cross-correlation coefficient, respectively, and $S_{\omega 2}$ represents a zero mean random (non-Gaussian) waveform with its spectrum centered about $2F_\omega$. By replacing ω with a everywhere in (14), it follows that translational acceleration cross products consist of a similar constant term and a series of the type S_{a2} centered about $2F_a$.

With the preceding description of aircraft motion it is possible to characterize the attitude error $[\tilde{B}]$ and the input acceleration error $\tilde{\mathbf{A}}$ in a straightforward manner. The attitude uncertainty consists of a skew-symmetric initial error matrix $[\tilde{B}_0]$ plus a skew-symmetric matrix with elements made up of time integrals of the total drift, delineated as follows (a complete analytical background of all drift sources is contained in Ref. 2): 1) steady drift sources (drift bias; interaction between mass unbalance and $\mathbf{F}$; net commutation error; anisoelastic rectification of translational vibration; angular rate rectification by gimbal displacement, anisoinertia, gyro loop mismatch, and angular acceleration about the output axis); 2) a series of the type $S_{\omega 1}$ [interaction of $\boldsymbol{\omega}$ with misalignment and scale factor errors, and instantaneous values of gyro delay error (including instantaneous effect of angle quantization) and of gyro output axis motion error]; 3) a mixed term, i.e., a series of the type $S_{\omega 1}$ with a secularly increasing amplitude, arising from the cross product of $\boldsymbol{\omega}$ with the secular portion of $\boldsymbol{\xi}$ in Eq. (2); 4) a series of the type $S_{\omega 2}$ (arising from the same angular rate products which produced rectification terms appearing in the steady drift); 5) a series of the type S_{a1} (interaction of $\mathbf{a}$ with mass unbalance; also, the anisoelastic cross product between $\mathbf{a}$ and $\mathbf{F}$); and 6) a series of the type S_{a2} (from anisoelastic product between components of $\mathbf{a}$).

The input translational acceleration error components can be delineated similarly, as follows (most of these errors can be deduced from Ref. 1, ignoring all terms containing products of error coefficients; the accelerometer separation error is derived in Appendix A): 1) fixed input acceleration error (interaction of $\mathbf{F}$ with misalignment and scale factor errors; accelerometer null bias, vibropendulous rectification of $\mathbf{a}$, rectification of $\boldsymbol{\omega}$ components by anisoinertia and by the centrifugal component of accelerometer separation error); 2) a series of the type $S_{\omega 1}$ (angular acceleration about the accelerometer output axis; tangential component of accelerometer separation error); 3) a series of the type $S_{\omega 2}$ (from the same sources as those producing fixed acceleration error through angular rate rectification); 4) a series of the type S_{a1} (interaction of $\mathbf{a}$ with misalignment and scale factor errors; pendulous offset error associated with cross product of $\mathbf{F}$ with $\mathbf{a}$; instantaneous effect of accelerometer delay error, including equivalent delay due to quantization); and 5) a series of the type S_{a2} (vibropendulosity).

At this point Eq. (9) could be rewritten with $[C]$ replaced by the right of (10), $\mathbf{A}$ replaced by $(\mathbf{F} + \mathbf{a})$, and with $[\tilde{B}]$ and $\tilde{\mathbf{A}}$ expressed as suggested previously. In principle, this result could then be expanded algebraically, followed by separation of terms into various categories (constant, secular, oscillatory, mixed, etc.) and, as explained previously, elimination of all terms (e.g., purely oscillatory error components with amplitudes comparable to coefficients of secularly increasing error sources) which make no appreciable contribution to the net total acceleration error. This process of eliminating insignificant terms is facilitated by the use of statistical averages and by comparisons between individual components of error. Accordingly, it will first be noted that all series having a form S_2 (i.e., either S_{a2} or $S_{\omega 2}$) can be neglected, because all products involving these will fall into one of the following three categories: 1) (Constant) $\times$ (S_2) which, by definition, has zero mean; 2) $(S_1) \times (S_2)$ which, when averaged, corresponds to a third central statistical moment; this always vanishes for a Gaussian random process; and 3) $(S_{\omega 1}) \times (S_2) \times (S_{a1})$, corresponding to a fourth central moment which, although not inherently negligible, is significantly smaller than other error terms which are retained.† It follows that the right side of (9) can be replaced by

$$[C]([\tilde{B}]\mathbf{A} + \tilde{\mathbf{A}}) \doteq [P]\{[I] + [Q]\}[\{[\tilde{B}_S] + [\tilde{B}_1]\} \times (\mathbf{F} + \mathbf{a}) + \tilde{\mathbf{A}}_0 + \tilde{\mathbf{A}}_\Omega + \tilde{\mathbf{A}}_a] \qquad (15)$$

where $[\tilde{B}_S]$ contains all steady and secular terms in $[\tilde{B}]$, $[\tilde{B}_1]$ consists of all terms in $[\tilde{B}]$ which involve series of the form S_{a1} or $S_{\omega 1}$; and $\tilde{\mathbf{A}}_0$, $\tilde{\mathbf{A}}_\Omega$, $\tilde{\mathbf{A}}_a$ represent the input acceleration errors which are constant, oscillatory with form $S_{\omega 1}$, and oscillatory with form S_{a1}, respectively. Immediately this expression can be rewritten, dropping all unrectified vibratory terms. Although some of the terms thus neglected have significant amplitudes (e.g., the amplitude of $\tilde{\mathbf{A}}_\Omega$ can be larger than $\tilde{\mathbf{A}}_0$), this effect will be overridden in the double integration of (9). The forcing function of (15) is therefore replaced by

$$[C]\{[\tilde{B}]\mathbf{A} + \tilde{\mathbf{A}}\} \doteq [P]\{[\tilde{B}_S]\mathbf{F} + [\tilde{B}_1]\mathbf{a} + [Q]([\tilde{B}_S]\mathbf{a} + [\tilde{B}_1]\mathbf{F}) + \tilde{\mathbf{A}}_0 + [Q](\tilde{\mathbf{A}}_\Omega + \tilde{\mathbf{A}}_a)\} \qquad (16)$$

An argument related to the preceding argument can be applied to the quantity $([\tilde{B}_1]\mathbf{a})$; whereas $[\tilde{B}_1]$ contains terms

† All fourth moments contain elements of $[Q]$ (which cannot exceed $\pi/180$), multiplied by integrals of oscillatory components from quadratic drift terms (which, due to integration, are divided by $2\pi F_a$ or $2\pi F_\omega$), as well as the drift coefficients themselves. The steady components from these same quadratic drift terms, by comparison, form secular elements in $[\tilde{B}]$ upon integration; the direct interaction of these secular attitude errors with $\mathbf{F}$ produces a secular acceleration error component which is not premultiplied by $[Q]$ nor divided by $2\pi F_a$ or $2\pi F_\omega$.

with fixed and secularly increasing amplitude, $[\tilde{B}_S]$ contains purely secular terms. Also, since the elements of $[Q]$ are of order $\pi/180$,

$$[Q]([\tilde{B}_S]\mathbf{a} + [\tilde{B}_1]\mathbf{F}) \ll [\tilde{B}_S]\mathbf{F} + [\tilde{B}_1]\mathbf{a} \tag{17}$$

It follows that

$$[C]([\tilde{B}]\mathbf{A} + \tilde{\mathbf{A}}) \doteq [P]\{[\tilde{B}_S]\mathbf{F} + \mathbf{A}_0 + \langle [Q](\tilde{\mathbf{A}}_\Omega + \tilde{\mathbf{A}}_\omega)\rangle\} \tag{18}$$

These latter simplifications depend upon $|\mathbf{a}|$ and $|\mathbf{F}|$ being of the same order of magnitude, but this is quite reasonable. In the last term, the angular brackets denote averaging, an obviously justifiable procedure at this point. The appendices show that this term is essentially equivalent to a rectified tangential ($\tilde{\mathbf{A}}_t$) and output axis motion ($\tilde{\mathbf{A}}_y$) error. Writing $[\tilde{B}_0]$ and $[N]t$ for the initial and secularly increasing attitude error matrices, respectively, Eqs. (9) and (18) combine to yield

$$\Delta\ddot{\mathbf{R}} + (g/R)\Delta\mathbf{R} = [P]\{[N]\mathbf{F}t + [\tilde{B}_0]\mathbf{F} + \tilde{\mathbf{A}}_0 + \tilde{\mathbf{A}}_t + \tilde{\mathbf{A}}_y\} \tag{19}$$

where, as previously explained, only the x (range) and y (cross-range) components of the solution are retained. It is convenient to write $\tilde{\mathbf{A}}_K$ for the sum $([\tilde{B}_0]\mathbf{F} + \tilde{\mathbf{A}}_0 + \tilde{\mathbf{A}}_t + \tilde{\mathbf{A}}_y)$ so that the forcing function is written simply as a secular term $[N]\mathbf{F}t$ plus a constant $\tilde{\mathbf{A}}_K$:

$$\Delta\ddot{\mathbf{R}} + (g/R)\Delta\mathbf{R} = [P]\{[N]\mathbf{F}t + \tilde{\mathbf{A}}_K\} \tag{20}$$

Analytical Results

Equation (20) shows that the range and cross-range navigation errors can be closely determined from uncoupled second-order differential equations with constant coefficients, having superimposed constant and linearly increasing forcing functions. In this section the solution is expressed directly in terms of familiar system and environmental parameters.

The coefficient (g/R) is immediately identified as the square of the Schuler rate ($W \doteq 2\pi$ rad/84 min) and the matrix $[P]$ is the result of a small yaw (φ_z) and pitch (φ_y) displacement:

$$[P] \doteq \begin{bmatrix} 1 & -\varphi_z & \varphi_y \\ \varphi_z & 1 & 0 \\ -\varphi_y & 0 & 1 \end{bmatrix} \tag{21}$$

The attitude error matrices consist of initial uncertainties in roll λ_1, pitch λ_2, and yaw λ_3 as well as corresponding secularly increasing drift angles:

$$[\tilde{B}_0] = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}$$

$$[N]t = \begin{bmatrix} 0 & -\langle n_3\rangle & \langle n_2\rangle \\ \langle n_3\rangle & 0 & -\langle n_1\rangle \\ -\langle n_2\rangle & \langle n_1\rangle & 0 \end{bmatrix} t \tag{22}$$

where $\langle n_j\rangle$ are the net drift rates defined in Ref. 2. They are recomputed here under somewhat modified conditions: 1) For cruise applications the roll, pitch, and yaw gyro spin axes are chosen parallel to the pitch, roll, and pitch axes, respectively. 2) The aircraft pitch oscillations are essentially uncoupled during cruise, i.e., $\langle\omega_1\omega_2\rangle = \langle\omega_2\omega_3\rangle = 0$.

It will be noted that the preceding two conditions eliminate the secular drifts due to gimbal displacement and anisoinertia, as well as certain other steady drift terms in individual channels. Effects of anisoelasticity, angular acceleration about the pitch axis, and mass unbalance are also minimized. With these two conditions and the previously defined dynamical environment, the methods derived in Ref. 2 apply directly to yield

$$\langle n_1\rangle = n_{d1} - \gamma_y\langle\omega_3^2\rangle - \gamma_{x1}F_x - \gamma_{xz}\langle a_1a_2\rangle \tag{23}$$

$$\langle n_2\rangle = n_{d2} + \gamma_y[\langle\omega_3^2\rangle - \langle\omega_1^2\rangle] + (\tau_1 - \tau_3)\langle\omega_1\omega_3\rangle + \rho_{\omega 13}(\omega_{3r} - \omega_{1r})\theta/2 + \gamma_{z2}F_x - \gamma_{xz}\langle a_1a_2\rangle \tag{24}$$

$$\langle n_3\rangle = n_{d3} + \gamma_y\langle\omega_1\omega_3\rangle - \gamma_{x3}F_z - \gamma_{xz}\langle a_2a_3\rangle \tag{25}$$

As mentioned in Ref. 2, misalignment errors in the presence of sustained unidirectional coning motions would also produce secular drifts. This component of drift rate would be proportional to products of rms angular rates and the associated coefficients of delayed (by $1/2\pi F_\omega$) correlation. Using Eq. (7) of Ref. 2 it can be rigorously shown that net scale factor errors of this type vanish by definition (because a narrow-band Gaussian waveform is uncorrelated with its own time integral), and rectification of misalignment error requires a significant degree of correlation between orthogonal rates in phase quadrature. Even under this questionable condition, the rectified misalignment error would be overshadowed by typical values of output axis angular acceleration drift. Consequently, this error source is excluded from the preceding formulation.

The components of $\tilde{\mathbf{A}}_0$ are determined from the fixed-input acceleration errors (which were listed and referenced in the preceding section) and from the directions of the accelerometer pendulous axes. By choosing each pendulous axis parallel to the corresponding gyro spin axis, the steady accelerometer anisoinertia error is eliminated here, and various other sources are minimized. Under this condition,

$$\tilde{\mathbf{A}}_0 = \begin{bmatrix} -\alpha_{z1}F_z + \mu_1F_x + \nu_1 - \beta_{xz}\langle a_1a_2\rangle \\ -\alpha_{z2}F_z + \alpha_{y2}F_x + \nu_2 - \beta_{xz}\langle a_1a_2\rangle \\ -\alpha_{z3}F_x + \mu_3F_z + \nu_3 - \beta_{xz}\langle a_2a_3\rangle \end{bmatrix} + \tilde{\mathbf{A}}_c \tag{26}$$

where $\tilde{\mathbf{A}}_c$ is defined in Appendix A. The differential equations for range and cross-range error are now written as

$$\begin{bmatrix} \Delta\ddot{X} \\ \Delta\ddot{Y} \end{bmatrix} + W^2\begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} = \begin{bmatrix} (\tilde{F}_1 - \varphi_z\tilde{F}_2 + \varphi_y\tilde{F}_3)t + \tilde{A}_{K1} - \varphi_z\tilde{A}_{K2} + \varphi_y\tilde{A}_{K3} \\ (\tilde{F}_2 + \varphi_z\tilde{F}_1)t + \tilde{A}_{K2} + \varphi_z\tilde{A}_{K1} \end{bmatrix} \tag{27}$$

where

$$\tilde{F}_1 = \langle n_2\rangle F_z \qquad \tilde{F}_2 = \langle n_3\rangle F_x - \langle n_1\rangle F_z \qquad \tilde{F}_3 = -\langle n_2\rangle F_x \tag{28}$$

and

$$\tilde{\mathbf{A}}_K = \begin{bmatrix} \mu_1F_x + (\lambda_2 - \alpha_{z1})F_z + \nu_1 - \beta_{xz}\langle a_1a_2\rangle \\ (\lambda_3 + \alpha_{y2})F_x - (\alpha_{z2} + \lambda_1)F_z + \nu_2 - \beta_{xz}\langle a_1a_2\rangle \\ -(\alpha_{z3} + \lambda_2)F_x + \mu_3F_z + \nu_3 - \beta_{xz}\langle a_2a_3\rangle \end{bmatrix} + \tilde{\mathbf{A}}_c + \tilde{\mathbf{A}}_t + \tilde{\mathbf{A}}_y \tag{29}$$

where $\tilde{\mathbf{A}}_t$ and $\tilde{\mathbf{A}}_y$ are defined in the appendices. Since initial translation errors necessitate only a trivial extension, the solution to (27) will be given here for the case

$$\Delta X(0) = \Delta Y(0) = \Delta\dot{X}(0) = \Delta\dot{Y}(0) = 0 \tag{30}$$

so that

$$\begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} = \frac{1}{W^2}\left\{\begin{bmatrix} \tilde{F}_1 - \varphi_z\tilde{F}_2 + \varphi_y\tilde{F}_3 \\ \tilde{F}_2 + \varphi_z\tilde{F}_1 \end{bmatrix}\left(t - \frac{\sin Wt}{W}\right) + \begin{bmatrix} \tilde{A}_{K1} - \varphi_z\tilde{A}_{K2} + \varphi_y\tilde{A}_{K3} \\ \tilde{A}_{K2} + \varphi_z\tilde{A}_{K1} \end{bmatrix}(1 - \cos Wt)\right\} \tag{31}$$

Navigation performance for a hypothetical set of parameters is illustrated in the next section.

Numerical Example

Equation (31) and the supporting relations were derived for a general set of conditions, for two reasons. First, by considering early portions of the flight (when F_x is appreciable), the navigation errors associated wtih motion along the flight path (e.g., heading errors and certain accelerometer scale factor and misalignment errors) can be ascertained.‡ Secondly, the analytical results, as presented in the preceding section, are not heavily dependent upon relative magnitudes of parameters (which will vary with future state of the art).

Since application of the analysis is quite straightforward, the conditions adopted for illustration can be restricted here in the interest of brevity. For straight and level cruise with negligible initial attitude uncertainty,

$$\varphi_y = \varphi_z = \lambda_1 = \lambda_2 = \lambda_3 = 0 \qquad F_x = O_g \qquad (32)$$

and the navigation error equations reduce to

$$\begin{bmatrix}\Delta X\\ \Delta Y\end{bmatrix} = \frac{1}{W^2}\left\{\begin{bmatrix}\langle n_2\rangle\\ -\langle n_1\rangle\end{bmatrix}F_z\left(t - \frac{\sin Wt}{W}\right) + \begin{bmatrix}A_{K1}\\ A_{K2}\end{bmatrix}(1 - \cos Wt)\right\} \qquad (33)$$

where the drift rates are again taken from (23) and (24), and

$$\begin{bmatrix}A_{K1}\\ A_{K2}\end{bmatrix} = \begin{bmatrix}-\alpha_{z1}\\ -\alpha_{z2}\end{bmatrix}F_z + \begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} - \begin{bmatrix}\beta_{xz}\\ \beta_{xz}\end{bmatrix}\langle a_1a_2\rangle + \tilde{\mathbf{A}}_c + \tilde{\mathbf{A}}_t + \tilde{\mathbf{A}}_y \qquad (34)$$

This latter expression will first be investigated for the set of coefficients in Table 1 (values are within state of the art, particularly if partial compensation may be considered) in an aircraft environment defined by a 1 g lift F_z and rms roll, pitch, and yaw rates of 0.004, 0.02, and 0.01 rad/sec, respectively. Straightforward computations show that, for rms translational vibration in the vicinity of 1 g, all of the sources in (34) produce errors of order (3×10^{-5} g). From (33) this produces a steady-state position error in the neighborhood of 0.1 naut mile plus a Schuler oscillation of the same amplitude. It is observed that

1) The error sources peculiar to strapdown systems [note in particular the last three terms of (34)] have not been proven detrimental thus far, in comparison to the errors that would be present even in platform applications.

2) If the remaining errors in the system (due to attitude drift) were no more serious than this, realistic performance evaluation would call for lifting some of the simplifications in (32). Other sources not included previously (e.g., interaction of accelerometer scale factor error with motion along the flight path) would then appear, affecting both the total navigation error and the relative magnitudes of range and cross-range error. It will be seen, however, that the navigation errors associated with attitude drift are larger by comparison.

Superimposed upon the navigation errors derived previously are the secular errors and Schuler oscillations associated with attitude reference drift. Table 2 represents a parameter set which, aside from the optimistic drift bias, can be obtained by present state of the art in the dynamical environment previously indicated. Assuming tight coupling for $\langle a_1a_2\rangle$ and $\langle\omega_1\omega_3\rangle$, the drift equations show that gyro loop mismatch

Table 1 Accelerometer error coefficients[a]

Misalignment (α)	3×10^{-5}
Accelerometer bias (ν)	2×10^{-5}
Vibropendulosity (β_{xz})	2×10^{-5}
Accelerometer size (r_{nm})	2 in.
Output axis rotation sensitivity (β_y)	0.003

[a] All units are given in Nomenclature unless otherwise noted.

‡ By using navigation errors thus derived to replace (30) as initial conditions for cruise, various error expressions can be obtained and superimposed upon cruise analysis results of the type illustrated here.

Table 2 Attitude reference error coefficients[a]

Drift bias (n_d)	10^{-7}
Output axis rotation sensitivity (γ_y)	10^{-3}
Gyro loop delay mismatch ($\tau_1 - \tau_3$)	2×10^{-4}
Angular resolution (θ)	2^{-14}
Gyro anisoelastic parameter (γ_{xz})	10^{-7}

[a] All units are given in Nomenclature unless otherwise noted.

effect is relatively low and all other sources produce drift rates of order 1.5×10^{-7} sec^{-1} (note: 10^{-7} rad/sec. is equivalent to 1.24 min of arc/hr). By (33) this corresponds to navigation errors of 2 naut miles/hr.

Although the example chosen was somewhat simplified, it does cover the major sources of error under present state-of-the-art conditions. It should be noted that commutativity is a major source in this example, even with an angular resolution of 2^{-14} rad, when yaw and roll rates are highly correlated.

Conclusion

An extensive analysis of navigation errors has been presented for gimballess inertial systems in cruising aircraft. With realistic aircraft motions taken into account (including complex lateral and angular vibrations, with coupling between axes), navigation errors can be expressed directly in terms of familiar system coefficients and dynamic environmental parameters. At the present state of the art, the major error sources are covered by Eqs. (23, 24, 33, and 34) under typical cruise conditions.

Appendix A: Accelerometer Separation Error

A noncoincident triad of accelerometers will measure components of three different vectors. It can be shown that the relative acceleration between two points separated by a vector $\mathbf{r}_m$ (fixed in body co-ordinates) is

$$\ddot{\mathbf{r}}_m = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_m) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_m$$

and, from consideration of the noncoincident accelerometer triad, the difference between the observed acceleration "vector" and the true acceleration of an arbitrary sensor package reference point is

$$\begin{bmatrix}\mathbf{i}\cdot\ddot{\mathbf{r}}_1\\ \mathbf{j}\cdot\ddot{\mathbf{r}}_2\\ \mathbf{k}\cdot\ddot{\mathbf{r}}_3\end{bmatrix} = \begin{bmatrix}(\boldsymbol{\omega}\cdot\mathbf{r}_1)\omega_1 - r_{11}\omega^2\\ (\boldsymbol{\omega}\cdot\mathbf{r}_2)\omega_2 - r_{22}\omega^2\\ (\boldsymbol{\omega}\cdot\mathbf{r}_3)\omega_3 - r_{33}\omega^2\end{bmatrix} + \begin{bmatrix}\dot{\omega}_2r_{13} - \dot{\omega}_3r_{12}\\ \dot{\omega}_3r_{21} - \dot{\omega}_1r_{23}\\ \dot{\omega}_1r_{32} - \dot{\omega}_2r_{31}\end{bmatrix}$$

where (**i**, **j**, **k**) represent the unit body x, y, z vectors, respectively; and (r_{mn}) is the (nth) component of the displacement vector $(\mathbf{r}_m)$ between the (mth) accelerometer and the sensor package reference point. To evaluate the resulting contribution to the navigation error forcing function, the preceding vector is transformed into reference coordinates by the matrix $[C]$ of Eq. 10; the result after averaging is (again, only the average errors are of interest, since these will overshadow the effects of the accompanying oscillatory components):

$$[P](\tilde{\mathbf{A}}_c + \tilde{\mathbf{A}}_t) \triangleq [P]\begin{bmatrix}\langle(\boldsymbol{\omega}\cdot\mathbf{r}_1)\omega_1\rangle - r_{11}\langle\omega^2\rangle\\ \langle(\boldsymbol{\omega}\cdot\mathbf{r}_2)\omega_2\rangle - r_{22}\langle\omega^2\rangle\\ \langle(\boldsymbol{\omega}\cdot\mathbf{r}_3)\omega_3\rangle - r_{33}\langle\omega^2\rangle\end{bmatrix} + [P]\left\langle[Q]\begin{bmatrix}\dot{\omega}_2r_{13} - \dot{\omega}_3r_{12}\\ \dot{\omega}_3r_{21} - \dot{\omega}_1r_{23}\\ \dot{\omega}_1r_{32} - \dot{\omega}_2r_{31}\end{bmatrix}\right\rangle$$

The first term $\tilde{\mathbf{A}}_c$ will be recognized as the centrifugal error. The second term is the tangential component of accelerometer separation error, rectified by the actual angular vibration as follows: Equation (11) has shown that the elements of $[Q]$ are proportional to time integrals of the angular rates.

When these are multiplied by corresponding time derivatives, the expressions given in (13) show that the result is proportional to mean quadratic angular rate terms. (Combination of the phase lag and phase lead will affect the algebraic sign of the product, but no appreciable reduction in correlation will be introduced.)

The complete expression can easily be derived from the relations indicated; for brevity of this presentation it suffices to note that 1) both the centrifugal and the rectified tangential error are of the same order of magnitude; 2) the accelerometer placement vectors ($\mathbf{r}_m$) cannot be chosen to cancel the mean separation errors, except in the improbable event of a consistent, predictable direction of the angular oscillation axis; and 3) the total net separation error is on the order of $\langle\omega^2\rangle$ multiplied by the accelerometer size (assuming of course that they are spaced as closely as possible; note that a two-degree-of-freedom accelerometer would be useful for this purpose).

Appendix B: Rectification of Remaining Oscillatory Acceleration Errors

The preceding Appendix briefly demonstrated, in connection with tangential acceleration error, how an oscillatory sensing error can be converted into a steady forcing function through the orthogonal transformation into reference coordinates. This phenomenon occurs repeatedly in the strapdown inertial navigation system; the immediate purpose here is to evaluate the same effect for the term with angular brackets in Eq. (18).

Considering first the quantity $\langle[Q]\tilde{\mathbf{A}}_a\rangle$, it is recalled that $\tilde{\mathbf{A}}_a$ is the result of 1) interaction of $\mathbf{a}$ with misalignment and scale factor errors, 2) pendulous offset error associated with the interaction between $\mathbf{F}$ and $\mathbf{a}$, and 3) imperfect synchronization between translational and rotational data (including random delay due to integrating accelerometer quantization). The first of these items can be ignored because it is overshadowed by a similar term in $\mathbf{A}_0$ (i.e., interaction between $\mathbf{F}$ with misalignment and scale factor errors) which is not premultiplied by the small elements of $[Q]$. A similar statement can be made regarding the second item (overshadowed by vibropendulosity). This leaves only the synchronization error; it will now be shown that this is also negligible because of low correlation between angular rate and perpendicular translational acceleration:

By analogy with the timing error analysis in Ref. 2 it is seen that $\tilde{\mathbf{A}}_a$ contains the delay interval multiplied by the derivative of the translational acceleration. Therefore $([Q]\tilde{\mathbf{A}}_a)$ will have a significant average value only if the time integrals of the angular rates correlate with the derivatives of the perpendicular§ translational accelerations. Equivalently, the angular rates themselves must correlate with the perpendicular lateral displacements; this, however is nearly opposite to the true situation. (With harmonic bending about the pitch axis, for example, the pitch angle is in phase with the yaw displacement; the pitch rate is therefore in phase quadrature.) Undoubtedly, some correlation between vehicle rate and orthogonal translation will be present with more complex vehicle motion (eq., delayed attitude control loop response to changes in orientation caused by wind gusts, etc.); however, the correlation between angle and orthogonal displacement will remain fairly tight. This will preclude the accumulation of a large average $\langle[Q]\tilde{\mathbf{A}}_a\rangle$, in comparison with the level of other errors present.

The remaining term $\langle[Q]\tilde{\mathbf{A}}_\Omega\rangle$ consists of the rectified tangential accelerometer separation error $\tilde{\mathbf{A}}_t$ defined in Appendix A, plus the net output axis motion error,

$$\tilde{\mathbf{A}}_y = \left\langle [Q]\begin{bmatrix} -\beta_y\dot{\omega}_3 \\ -\beta_y\dot{\omega}_3 \\ -\beta_y\dot{\omega}_1 \end{bmatrix}\right\rangle$$

for the previously described (noncyclic) accelerometer orientations. In this expression (β_y) is the ratio of the accelerometer output axis moment of inertia to the pendulosity.[1] Again, the complete expression can readily be derived, but it is sufficient here to note that the net error is proportional to $(\beta_y\langle\omega^2\rangle)$.

References

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[2] Farrell, J., "Performance of strapdown inertial attitude reference systems," *AIAA/JACC Guidance & Control Specialist Conference* (American Institute of Aeronautics and Astronautics, New York, August 1966); also J. Spacecraft Rockets **3,** 1340–1347 (1966).

[3] Pitman, G. R. (ed.), *Inertial Guidance* (John Wiley & Sons Inc., New York, 1962), p. 164.

§ Premultiplication by a skew-symmetric matrix is equivalent to the formation of a vector cross product.